Solutions to problem set #1

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1 Problem #1

In this problem, we were asked to compute λ_D, plasma parameter Λ, V_{te}, and ν_{ei}.

Credit was divided between applying an appropriate formula and actually getting "accurate" numbers.

$$\lambda_D \approx 704\sqrt{T(ev)/n(cm^3)} \tag{1}$$

$$\Lambda = n_0 \lambda_D^3 \tag{2}$$

$$V_{te} = \sqrt{\alpha k T/m} \tag{3}$$

where α takes on various values, depending on whose (Chen or Nicholson or whomever) definition of Vte was used. The numbers below used α=1, but other definitions were given full credit.

Even more controversial was the definition of ν_{ei}, with some people using the formula from Chen, some from Nicholson, and some the approximate expression. Answers were considered correct if they were within an order of magnitude or so. Here are some of the expressions considered valid: (n_e in cm^{-3})

$$\nu_{ei} \approx \frac{\omega_e}{\Lambda} \tag{4}$$

$$\nu_{ei} \approx \frac{8\pi n_0 e^4 \ln\Lambda}{m_e^2 v_2^3 \omega_e} \tag{5}$$

$$\nu_{ei} \approx \frac{2\times 10^{-6} Z n_e e^4 \ln\Lambda}{T_{ev}^{3/2}} \tag{6}$$

n	T_e	λ_D	Λ	V_{te}	ν_{ei}
cm^{-3}	eV	cm		m/s	s^{-1}
10^{18}	1000	2.34×10^{-5}	1.3×10^{4}	1.3×10^{7}	4.3×10^{9}
10	10	740	4.1×10^{9}	1.3×10^{6}	4.4×10^{-5}
10^{6}	0.1	0.23	1.2×10^{4}	1.3×10^{5}	4.7×10^{3}
10^{10}	100	0.074	4.1×10^{6}	4×10^{6}	1.4×10^{3}

2 Problem #2

We're given:

$$\frac{\partial E_z}{\partial z} = \frac{\delta e n_0}{\epsilon_0} \tag{7}$$

$$\omega_p = c k_z \tag{8}$$

$$|\frac{\partial E_z}{\partial z}| \approx k_z |E_z| \tag{9}$$

We also know:

$$\omega_p = \sqrt{\frac{e^2 n_0}{\epsilon_0 m}} \tag{10}$$

So we can immediately plug in:

$$|E_z| \approx \frac{1}{k_z}\frac{\delta e n_0}{\epsilon_0} = \frac{c}{\omega_p}\frac{\delta e n_0}{\epsilon_0} = c\delta\sqrt{\frac{m n_0}{\epsilon_0}} \tag{11}$$

Substituting in the numbers $n_0 = 10^{25}$ and $\delta = 0.5$ gives 1.5×10^{11} V/m.

3 Problem #3 (Nicholson 1.4)

The most successful approach to this problem was to assume a potential, use the boltzmann relationship, multiply the suggested gravitational density dependence times the boltzmann relation, equate the two densities of each species using quasi-neutrality, equate the exponents, solve for ϕ, like so:

$$n_{e,i} = n_0 e^{-m_{e,i} g z/kT} \tag{12}$$

Multiply this expression by the Boltzmann relation, yielding:

$$n_e = n_0 e^{(e\phi - m_e g z)/kT} \tag{13}$$

$$n_i = n_0 e^{(-e\phi - m_i g z)/kT} \tag{14}$$

We were also told to use $n_e \approx n_i$, and if that is true, then the exponents are even more approximately equal. Equating the exponents:

$$e\phi - m_e g z = -e\phi - m_i g z \tag{15}$$

yields:

$$\phi = \frac{(m_e - m_i) g z}{2e} \tag{16}$$

which when substituted back gives a density:

$$n = n_i = n_e = n_0 e^{-\frac{(mi+me)}{2}\frac{gz}{kT}} \tag{17}$$

which is self consistent.

No one made a decent go at getting good boundary conditions: the solution generated here requires a constant electric field throughout the plasma, and therefore an infinite potential. However, Nicholson also gives us no boundary conditions, and I can't think of a better solution, so the one given here received full credit.

4 Problem #4

In this problem we were asked to justify the "order of magnitude" scalings in Nicholson for ν_{ee}, ν_{ii}, and ν_{ie}. Rather than repeat the entire derivation for the various cases, most people made various arguments pointing out why the original derivation for $\nu_e i$ yielded the scalings given in Nicholson. Given here are the arguments I considered most convincing from the homework solutions submitted:

For ν_{ee}, consider doing the electron-electron collision in the center of mass frame. This transformation introduces a reduced mass, which is a factor of $\sqrt{2}$, and otherwise does not change the original derivation, so ν_{ee} is of the order of ν_{ei}.

For ν_{ii}, we simply take the result for $\nu_{ee} \approx \nu_{ei}$, note that the expression $\nu_{ee} \propto m_e^{-1/2}$. Forming a ratio:

$$\frac{\nu_{ee}}{\nu_{ii}} = \sqrt{\frac{m_i}{m_e}} \tag{18}$$

(This next solution is due to Erik Rosolowsky) For ν_{ie}: Considering ν_{ie}, we can simply look at ν_{ei} from the perspective of the ion. It feels the same force, but the impulse scales like the mass, so the $\Delta v_\perp$ will be scaled by $\frac{m_e}{m_i}$ causing the ion to require that factor more scatters to reach our arbitrary collision angle: therefore, $\nu_{ie} = \frac{m_e}{m_i}\, \nu_{ei}$.

5 Problem #5

We were given two equations relating temperatures parallel and perpendicular to the magnetic field:

$$\frac{dT_{\|}}{dt} = 2\nu(T_{\perp} - T_{\|}) \tag{19}$$

$$\frac{dT_{\perp}}{dt} = \nu(T_{\|} - T_{\perp}) \tag{20}$$

For part (a), we were asked to show these equations conserved energy. So we verify

$$\frac{dT_{\|}}{dt} + 2\frac{dT_{\perp}}{dt} =^{?} 0 \tag{21}$$

with the factor of 2 in front of the second term coming in because of the 2 degrees of freedom in the perpendicular direction. You do indeed get 0 if you perform the algebra.

For part (b), the easiest approach was to subtract the two equations given, yielding:

$$\frac{d(T_{\|} - T_{\perp})}{dt} = 3\nu(T_{\|} - T_{\perp}) \tag{22}$$

and substituting $\Delta T = (T_{\|} - T_{\perp})$ to yield:

$$\frac{d\Delta T}{dt} = 3\nu\Delta T \tag{23}$$

which yields a time constant $\tau = 1/3\nu$ by inspection.

For part (c), you put the numbers 1eV and $n = 10^{16}m^{-3}$ into

$$\nu \approx \frac{ne^4}{2\pi\epsilon_0^2 m^{1/2}(kT)^{3/2}} \ln\Lambda \tag{24}$$

Plugging in gave: $\tau = 1.8 \times 10^{-7}s$.

6 Problem #6 (nicholson 2.1)

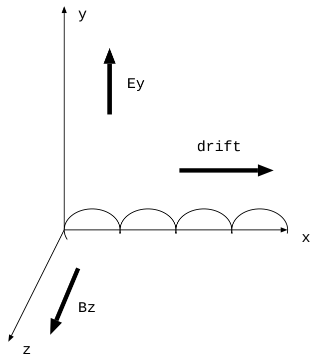

Figure 1: Particle orbit

For part (a), the important feature was the direction of the drift. Of lesser importance, but still assigned some credit, was the shape of the orbit. Some people drew the orbit as looping back on itself, but it won't with the initial conditions given.

For part (b), we were asked to derive the actual orbit of the particle. Start with the lorentz force:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{25}$$

We can write this as three scalar equations:

$$x : 0 + v_y B_z = \frac{m}{q}\dot{v}_x \tag{26}$$

$$y : E_0 - v_x B_z = \frac{m}{q}\dot{v}_y \tag{27}$$

$$z : 0 - 0 = \frac{m}{q}\dot{v}_z \tag{28}$$

Take the first time derivative of the equation for y:

$$\dot{v}_x B_z = -\frac{m}{q}\ddot{v}_y \tag{29}$$

$$\dot{v}_x = -\frac{m}{qB_z}\ddot{v}_y \tag{30}$$

substituting into the equation for x:

$$\ddot{v}_y = -\frac{q^2 B_z^2}{m^2} v_y \tag{31}$$

Which has the general oscillatory solution

$$v_y = A \sin \Omega t + C \cos \Omega t \tag{32}$$

where

$$\Omega \equiv \frac{q B_z}{m} \tag{33}$$

Note that at t=0, $v_y = 0$, so therefore, $C = 0$,

$$v_y = A \sin \Omega t \tag{34}$$

Integrating v_y,

$$y(t) = y_0 - \frac{A}{\Omega} \cos \Omega t \tag{35}$$

We know $y(0) = 0$, so

$$y(t) = \frac{A}{\Omega}(1 - \cos \Omega t) \tag{36}$$

Now, taking the equation for v_y and substituting it into equation for y:

$$E_0 - v_x B_z = \frac{m}{q} A \Omega \cos \Omega t \tag{37}$$

$$v_x = \frac{E_0}{B_z} - \frac{m}{q B_z} A \Omega \cos \Omega t \tag{38}$$

Applying the initial condition v_x=0 at t=0, we find that

$$A = \frac{E_0}{B_z} \tag{39}$$

so that:

$$v_x = \frac{E_0}{B_z}(1 - \cos \Omega t) \tag{40}$$

Integrating (and applying the initial condition) to get $x(t)$:

$$x(t) = \frac{E_0}{B_z}(t - \frac{\sin \Omega t}{\Omega}) \tag{41}$$

And substituting A into the equation for $y(t)$:

$$y(t) = \frac{E_0}{B_z \Omega}(1 - \cos \Omega t) \tag{42}$$

For part c, all that is required is to group all the sin/cos terms together and call that the oscillatory part, and note that what is left is a constant drift. Since the drift is constant, after averaging the oscillatory portions (which go to zero), there is no net acceleration. Matthew Allen explained why there is no net acceleration this way: “The average drift force $qE_0\hat{y}$ is balanced by the average Lorentz force due to drift velocity:

$$q(\mathbf{v} \times \mathbf{B}) = q\frac{E_0}{B_0}\hat{x} \times B_0\hat{z} = -qE_0\hat{y} \tag{43}$$

.”

Part (d) was a little tricky. The drift velocty $\frac{E_0}{B_0}$ is independent of charge and mass of the particle, so particles of both charges flow in the same direction at the same velocity. In a CHARGE NEUTRAL plasma, this would mean no net current would flow due to this drift.

For part (e), this is the generic force... On can simply substitute in F/q for E in the E cross B drift.

$$\mathbf{v}_d = \frac{\mathbf{F} \times \mathbf{B}_0}{qB_0^2} \tag{44}$$

7 Problem #7

The key to solving Nicholson 2.3 was to follow the instructions given as robotically as possible.

Given an electric field $\mathbf{E} = -\dot{E}t\hat{y}$, you have an E cross B drift velocity:

$$v_d = -\frac{\dot{E}t}{B_0} \tag{45}$$

The “robotic” step is that one doesn’t take a time derivative of the quantity $\dot{E}$, which is intended to be time-independent: the time dependence goes in the t: taking the derivative and substituting:

$$F = ma = m\dot{v} = -\frac{m\dot{E}}{B_0} \tag{46}$$

Plugging this force into the equation

$$v_d = \frac{c}{q}\frac{\mathbf{F}_\perp \times \mathbf{B}}{B^2} \tag{47}$$

yields equation 2.41.

8 Problem #8

For part a, we're given a magnetic mirror with a mirror ratio of 2. The loss cone is defined by:

$$\sin \Theta < \frac{1}{\sqrt{R}} \tag{48}$$

which works out to be an angle of 45 degrees. Here is a picture of the situation:

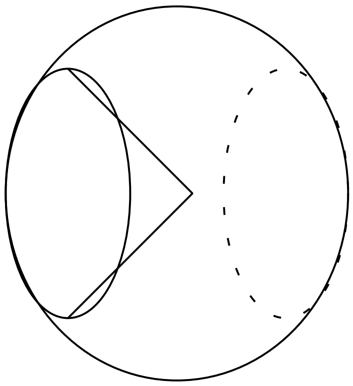

Figure 2: Mirror loss cone

Particles with velocities in the cone left: particles with velocities outside the cone stay, at least initially before collisions. There are various ways to compute the actual fraction, but I hope people find it intuitive that the fraction staying should be larger than 0.5.

The actual fraction can be evaluated by performing this integral:

$$\int_0^r \int_0^{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 \sin\phi dr d\theta d\phi \tag{49}$$

and yields a fraction of $\frac{1}{\sqrt{2}}$.

For part b), only crude estimates could be made.

(b) (1), loss of unconfined electrons: since the device is one meter, and the electrons simply fly out, $1m/v_{te}$ gives the characteristic time of approx. 5×10^{-8} sec.

(b) (2), loss of unconfined ions: since the device is one meter, and the ions simply fly out, $1m/v_{ti}$ gives the characteristic time of approx. 2×10^{-6} sec.

(b) (3), loss of initially confined electrons: for this, we need to know the scale time for how long it takes for the electrons to be scattered into the loss cone. $\tau \approx 1/\nu ei$ gives the characteristic time of approx. 1×10^{-4} sec. Electron-ion collisions should dominate, because many electrons will have left. All of the electrons will not leave because of the unneutralized space charge of the ions: a potential will build up which will tend to retain electrons.

(b) (4), loss of initially confined ions: for this, we need to know the scale time for how long it takes for the ions to be scattered into the loss cone. $\tau \approx 1/\nu ii$ gives the characteristic time of approx. 1×10^{-3} sec.

For fusion processes, we care about the confinement time of the ions: they're the particles doing the fusing.

9 Problem #9

This problem was most readily done via conservation of action, or equivalently by using the "bounce" adiabatic invariant:

$$\int_a^b v_{\parallel} \cdot ds = J = const \tag{50}$$

Another way to see it is that phase-space area is conserved:

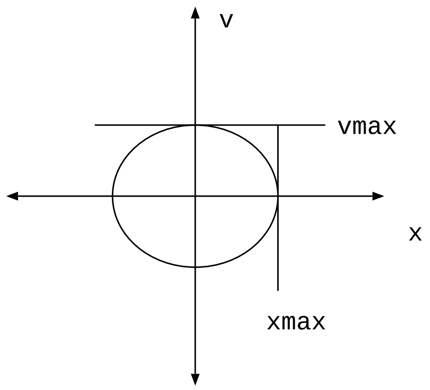

Figure 3: Phase space diagram

$$xmax = A \tag{51}$$

$$Kemax = \alpha A^2 \tag{52}$$

$$vmax = \sqrt{\frac{2\alpha A^2}{m}} \tag{53}$$

$$J = area = \pi vmaxxmax = \pi A^2 \sqrt{\frac{2\alpha_0}{m}} \tag{54}$$

Setting up a proportionality, we see that:

$$A_{new}^2 \sqrt{\alpha_1} = A^2 \sqrt{\alpha_0} \tag{55}$$

so

$$A_{new} = A \left(\frac{\alpha_0}{\alpha_1} \right)^{1/4} \tag{56}$$